# Non-Vanishing and Orthogonal Basis of Symmetry Classes of Tensors

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**Abstract**

By Cayley's theorem, any finite group $G$ of order $n$ can be regarded as a subgroup of the symmetric group $\$_n$. Let $\chi$ be any irreducible complex character of $G$ and let $V_\chi^n(G)$ denote the symmetry classes of tensors associated with $G$ and $\chi$. In this paper assuming the Cayley representation of $G$, we obtain a formula for the dimension of $V_\chi^n(G)$ and discuss its non-vanishing in general. A necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ is also obtained.

**Keywords:** Symmetry class of tensors, Decomposable symmetrized tensor, Orthogonal basis, Cayley representation.

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## 1 Introduction

Let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{n}{\otimes}V$ be the $n$th tensor power of $V$ and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation $g$ in $\$_n$, there corresponds a unique linear operator $P(g) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(g)(v_1 \otimes \cdots \otimes v_n) = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)}$. Let $G$ be a subgroup of $\$_n$ and $\mathrm{I}(G)$ the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{T(G,\chi) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V \Big| \ T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)P(g), \ \chi \in \mathrm{I}(G)\right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under $T(G,\chi)$ is called the *symmetry class of tensors* associated with $G$ and $\chi$ and is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \cdots * v_n$ and is called a *decomposable tensor*. It is well known that

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)m^{c(g)}, \tag{1}$$

where $c(g)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of $g$ (see [7]). Also,

$$\bigotimes^{n} V = \bigoplus_{\chi \in \mathrm{I}(G)} V_\chi^n(G) \tag{2}$$

is a direct sum.

Let $\Gamma_m^n$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leq \alpha_i \leq m$ so that $\alpha$ is a mapping from a set of $n$ elements into a set of $m$ elements. Then the group $G$ acts on $\Gamma_m^n$ by $g.\alpha := (\alpha_{g^{-1}(1)}, \ldots, \alpha_{g^{-1}(n)})$, where $g \in G$ is a permutation on $n$ letters and $\alpha \in \Gamma_m^n$ is a mapping from a set of $n$ elements into a set of $m$ elements. Therefore, the action may be written as $g.\alpha = \alpha g^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{g.\alpha |\ g \in G\}$ be the *orbit* with representative $\alpha$, and also let $G_\alpha$ be the *stabilizer* of $\alpha$, i.e., $G_\alpha = \{g \in G |\ g.\alpha = \alpha\}$. Let $\Delta$ be a system of distinct representatives of the orbits of $G$ acting on $\Gamma_m^n$ and define

$$\overline{\Delta} = \Big\{\alpha \in \Delta |\ \sum_{g \in G_\alpha} \chi(g) \neq 0\Big\},$$

and let $\hat{\Delta}$ be the union of those equivalence classes represented by elements of $\overline{\Delta}$.

Let $\{e_1, \ldots, e_m\}$ be a basis of $V$. Denote by $e_\alpha^*$ the tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$. For $\gamma \in \overline{\Delta}$, $V_\gamma^* = \langle e_{g.\gamma}^* |\ g \in G\rangle$ is called the *orbital subspace* of $V_\chi^n(G)$. It follows that

$$V_\chi^n(G) = \bigoplus_{\gamma \in \overline{\Delta}} V_\gamma^*, \tag{3}$$

is a direct sum. In [4], Freese proved that

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g), \tag{4}$$

in particular, if $\chi$ is of degree one, then $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$.

A particular case appears when we assume that $V$ is an $m$-unitary space. In this case, the inner product on $V$ induces an inner product on $\overset{n}{\otimes} V$ whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \cdots * u_n \mid v_1 * \cdots * v_n \rangle = \frac{\chi(1)}{|G|} d_\chi^G(A),$$

where $A = [a_{ij}]_{n\times n} = [\langle u_i|v_j\rangle]_{n\times n}$ and $d_\chi^G(A) = \sum_{g\in G}\chi(g)a_{1g(1)}\dots a_{ng(n)}$ is the *generalized matrix function.*

With respect to the above inner product, the sums that appeared in (2) and (3) are orthogonal direct sums. Also, if $\{e_1,\dots,e_m\}$ is an orthonormal basis of $V$, then we obtain

$$\langle e_\alpha^* \mid e_\beta^*\rangle = \begin{cases} \frac{\chi(1)}{|G|}\sum\limits_{\sigma\in G_\beta}\chi(\sigma\tau^{-1}) & \text{if} \quad \alpha=\tau.\beta \text{ for some } \tau\in G,\\ \\ 0 & \text{if} \quad O(\alpha)\neq O(\beta). \end{cases}$$

In particular, by taking the norm of $e_\alpha^*$, with respect to the induced inner product, one can easily obtain the condition $e_\alpha^*\neq 0$ if and only if $\alpha\in\hat{\Delta}$.

If $\alpha = g.\gamma$ and $\beta = g'.\gamma$, then $gg'^{-1}.\beta=\alpha$, and so if we let $\tau = gg'^{-1}$ and use the above formula for $\langle e_\alpha^* \mid e_\beta^*\rangle$, then we obtain

$$\langle e_{g.\gamma}^* \mid e_{g'.\gamma}^*\rangle = \frac{\chi(1)}{|G|}\sum_{\sigma\in G_\gamma}\chi(g'\sigma g^{-1}). \tag{5}$$

An orthogonal basis of the form $\{e_\alpha^*|\ \alpha\in S\}$, where $S$ is a subset of $\Gamma_m^n$, is called an *orthogonal basis of decomposable symmetrized tensors* for $V_\chi^n(G)$. By (3), $V_\chi^n(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if, for all $\gamma\in\overline{\Delta}$, the orbital subspace $V_\gamma^*$ has an orthogonal basis of decomposable symmetrized tensors. In particular, if $\chi$ is of degree one, since $\dim V_\gamma^* = 1$ for all $\gamma\in\overline{\Delta}$, then $V_\gamma^*$ has an orthogonal basis of decomposable symmetrized tensors for all $\gamma\in\overline{\Delta}$ which implies that $V_\chi^n(G)$ has such a basis. Non-vanishing of $V_\chi^n(G)$ were studied by several authors and they found a formula for $\dim V_\chi^n(G)$ in a more closed form than (1). Also, the existence of an orthogonal basis of decomposable symmetrized tensors for these vector spaces was considered (see, for example, [1, 14]). In [8] and [10], a formula for $\dim V_\chi^n(G)$ is also given when $G$ is equal to the whole group $\$_n$, and in [13], a formula for calculating $\dim V_\chi^n(G)$ is given in the case that $G = \langle\pi_1\rangle\cdots\langle\pi_p\rangle$ and in [2] for $G=\langle\pi_1\dots\pi_p\rangle$ is given, where $\pi_i$'s, $1\le i\le p$, are disjoint cycles in $\$_n$.

Also, in [5] a necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ is given, when $G$ is a cyclic or a dihedral group and in [3] when $G$ is a dicyclic group.

In this paper, we let $G$ be a subgroup of $\$_n$ by acting faithfully on a set of $n$ elements.

In this case, the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in \mathrm{I}(G)$ and we will discuss the non-vanishing property of these vector spaces. As a special case, that is, when $G$ is a group of order $n$ and acts on $G$ by right multiplication, we can prove that, for all $\chi \in \mathrm{I}(G)$, $V_\chi^n(G) \neq 0$, and we will find a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for these vector spaces.

## 2 Main Results

Let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$ and let $G$ be a finite group and $\Omega$ a set of $n$ elements. Suppose $G$ acts faithfully on $\Omega$, so we can assume that $G$ is a subgroup of $\$_n$, i.e., $G = \{g|\ g \in G\} = \{\sigma_g|\ g \in G\}$, where $\sigma_g :\ \Omega \to \Omega$ defined by $\sigma_g(\omega) = g.\omega$ for all $\omega \in \Omega$, is a permutation on $n$ letters. Therefore, the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in \mathrm{I}(G)$. $G$ as a subgroup of $\$_n$ acts on $n$ letters and we denote by $\theta$ its permutation character. For $g \in G$, the value $\theta(g)$ is the number of letters fixed by $g$, i.e., the number of cycles of length one in the cycle structure of $g$. In the following lemma we give a formulation of (1) in terms of $\theta$.

**Lemma 1** *Let $G$ be a finite group and $\Omega$ a set of $n$ elements. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $\chi \in \mathrm{I}(G)$, we have*

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) m^{\left(\theta\downarrow_{\langle g\rangle}, 1_{\langle g\rangle}\right)_{\langle g\rangle}}.$$

*Proof.* Suppose $g = \sigma_g \in G$. Then by definition, $c(g)$ is the number of cycles in the cycle structure of $g$ including cycles of length one. But it is easy to see that this number is equal to the number of orbits of the cyclic group $\langle g\rangle$ acting on $\Omega$. By Burnside's Lemma (see [9, p. 59]) the number of orbits of $\langle g\rangle$ acting on $\Omega$ is $\frac{1}{|\langle g\rangle|}\sum_{\sigma \in \langle g\rangle} \theta(\sigma)$ and so $c(g) = \left(\theta \downarrow_{\langle g\rangle}, 1_{\langle g\rangle}\right)_{\langle g\rangle}$, hence the theorem follows by (1). $\square$

As an application of the above lemma, we give here a solution to problem 5.18 in [6] and obtain a number theoretical result.

**Proposition 2** *Let $G$ be a finite group. Write $a(k) = |\{g \in G|\ o(g) = k\}|$. Then the polynomial*

$$f(x) = \frac{1}{|G|} \sum_k a(k) x^{|G|/k}$$

*takes on integer values whenever $x \in \mathbb{Z}$.*

*Proof.* First, we claim that $\psi : G \to \mathbb{C}$, defined by $\psi(g) = (-1)^{|G|/o(g)}$, $g \in G$, is a generalized character of $G$. If $|G|$ is odd, then $\psi(g) = -1$, for all $g \in G$. This leads to $\psi = -1_G$ and so $\psi$ is a generalized character of $G$. Hence, we suppose $|G|$ is even. Consider $D : G \to GL_n(\mathbb{C})$ the regular representation of $G$, where $|G| = n$. Then by problem 2.3 of [6], $\det : G \to \mathbb{C}$, defined by $\det(g) = \det\big(D(g)\big)$, $g \in G$, is a linear character of $G$. But it is easy to see that

$$\det\big(D(g)\big) = (-1)^{\big(o(g)-1\big)|G|/o(g)}, \qquad g \in G.$$

Since $|G|$ is even, for all $g \in G$, we have

$$\det(g) = \det\big(D(g)\big) = (-1)^{|G|-|G|/o(g)} = (-1)^{|G|/o(g)} = \psi(g).$$

Therefore, $\det = \psi$ and so in this case, $\psi$ is a linear character of $G$, and our claim is established.

Now by the notation in lemma 1, we consider the function $P : g \to P(g)$. It can be easily shown that $P$ is a representation of $G$ affording the character $\phi_m$ such that

$$\phi_m(g) = m^{c(g)} = m^{\big(\theta\downarrow_{\langle g\rangle}, 1_{\langle g\rangle}\big)_{\langle g\rangle}}, \qquad g \in G.$$

Let $\Omega = G$ and consider that $G$ acts on $\Omega$ by right multiplication, and put $|G| = n$. In this case, $G$ can be regarded as a subgroup of $\$_n$, whose action on $\Omega$ is regular and therefore, $c(1) = n = |G|$ and $c(g) = |G|/o(g)$ for $g \neq 1$. Thereby, for any $\chi \in \mathrm{I}(G)$, we have

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) m^{|G|/o(g)}.$$

Now, taking $\chi$ to be the principal character of $G$, we obtain

$$\dim V_\chi^n(G) = \frac{1}{|G|} \sum_{g \in G} m^{|G|/o(g)},$$

which is a natural number. For all $x \in \mathbb{Z}$, we have

$$f(x) = \frac{1}{|G|} \sum_{k} a(k) x^{|G|/k} = \frac{1}{|G|} \sum_{g \in G} x^{|G|/o(g)}.$$

Therefore, if $x > 0$, then $f(x) = (\phi_x, 1_G)_G \in \mathbb{N}$ and if $x < 0$, then $f(x) = (\phi_{-x}, \psi)_G \in \mathbb{Z}$. This is because $\psi$ is a generalized character of $G$. Therefore, $f(x)$ takes on integer values, for all $x \in \mathbb{Z}$. $\square$

If $G$ is a cyclic group, then for any divisor $k$ of $|G|$ we have $a(k) = \varphi(k)$, where $\varphi$ is the Euler $\varphi$ -function, and therefore, Proposition 2 implies that

$$\frac{1}{|G|}\sum_{k||G|}\varphi(k)x^{|G|/k} \in \mathbb{Z},$$

for all $x \in \mathbb{Z}$. Hence, if $G$ is a cyclic group of order $p$ , $p$ prime , then $\frac{1}{p}(x^p + (p-1)x) \in \mathbb{Z}$ which implies Fermat's little theorem $x^p \overset{p}{\equiv} x$.

In [11] and [12], a similar consideration led the authors to obtain Fermat's little theorem as a consequence. Let $G$ be a subgroup of the symmetric group on a finite set $\Omega$ and let $H \leq G$. Let $\mu$ be a function defined on subgroups of $G$ recursively by $\mu(H, H) = 1$ and $\mu(H, T) = -\sum_{H\leq K<T}\mu(H, K)$. If $H = 1$ is the trivial subgroup of $G$, then it is easy to prove that $\mu(1, T) = \mu(|T|)$ is the ordinary Möbius function of number theory. For a subgroup $T$ of $G$, let $c(T)$ be the number of orbits of $T$ on $\Omega$. Then it is proved in [12] that, for any $a \in \mathbb{Z}$, we have

$$\frac{|H|}{|N_G(H)|}\sum_{T\geq H}\mu(H, T)a^{c(T)} \in \mathbb{Z}.$$

If we let $H$ be the trivial subgroup of $G$, then the above formula becomes

$$\frac{1}{|G|}\sum_{T}\mu(1, T)a^{c(T)} \in \mathbb{Z}.$$

Now, assume that $G$ is embedded in $\$_\Omega$ by Cayley representation. If $G$ is assumed to be a cyclic group, then we have

$$\frac{1}{|G|}\sum_{k||G|}\mu(k)x^{|G|/k} \in \mathbb{Z}.$$

Now, if $G = \mathbb{Z}_p$, $p$ prime , then by the above formula, we obtain $\frac{1}{p}(a^p - a) \in \mathbb{Z}$ and consequently, we re-obtain Fermat's little theorem.

In the following we obtain a result about the non-vanishing of the symmetry classes of tensors which is a generalization of the known results.

**Lemma 3** *Let $G$ be a finite group and $\Omega$ a set of $n$ elements, $n \geq 2$. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$.*

*If, for all $g \in G - \{1\}, |fix(g)| \leq l$, then for all $m \geq l + 2$ and for all $\chi \in \mathrm{I}(G)$, we have $V_\chi^n(G) \neq 0$ (fix$(g)$ denotes the set of fixed points of $g$ upon its action on $\Omega$).*

*Proof.* Suppose for all $g \in G - \{1\}$, $|\mathrm{fix}(g)| \leq l'$, where $l'$ is the sharp upper bound, i.e., there is a $g \in G - \{1\}$ such that $|\mathrm{fix}(g)| = l'$. Then, we have $l' \leq l$ and $l' \leq n - 2$. Consider the action of the group $G$ on $\Gamma_m^n$ and put $\gamma = (1, 2, \ldots, l', l'+1, l'+2, \ldots, l'+2)$. If $m \geq l+2$, then $m \geq l'+2$, but $l'+2 \leq n$, therefore $\gamma \in \Gamma_m^n$, and we can choose $\Delta$ such that $\gamma \in \Delta$. By our hypothesis, one easily obtains that $G_\gamma = \{1\}$, so $\sum_{g \in G_\gamma} \chi(g) = \chi(1) \neq 0$ and therefore, $\gamma \in \overline{\Delta}$. But by (4), we have

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) = \chi(1)^2 \neq 0,$$

therefore, $V_\gamma^* \neq 0$, and by (3), we have $V_\chi^n(G) \neq 0$. $\square$

Since, for all non-identity $g \in G$, we have $|\mathrm{fix}(g)| \leq n - 2$, so we obtain the following result (see [10]) which is a consequence of the above lemma.

**Corollary 4** *Let $G$ be a finite group and $\Omega$ a set of $n$ elements, $n \geq 2$. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $m \geq n$ and all $\chi \in \mathrm{I}(G)$, we have $V_\chi^n(G) \neq 0$. In particular, for the subgroup $G$ of $\$_n$, we have $V_\chi^n(G) \neq 0$, for all $m \geq n$.*

Now, we consider a special case. Suppose $G$ is a group of order $n$ and $G$ acts on $\Omega = G$ by right multiplication, i.e., for all $g \in G$, $w \in \Omega$, $g.w = gw$. This action is faithful, therefore, $G$ is a subgroup of $\$_n$ and we say $G$ is a subgroup of $\$_n$ by *Cayley representation.* By Lemmas 1 and 3, we re-obtain the following result.

**Theorem 5** *Let $G$ be a group of order $n$, that is, a subgroup of $\$_n$ by Cayley representation. If $V$ is an $m$-dimensional vector space over the complex field $\mathbb{C}$, then for all $\chi \in \mathrm{I}(G)$, we have*

$$\dim V_\chi^n(G) = \frac{\chi(1)}{n} \sum_{g \in G} \chi(g) m^{n/o(g)},$$

*in particular, for all $m \geq 2$, $V_\chi^n(G) \neq 0$.*

In closing, we give a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ in the case where the finite group $G$ of order $n$ is embedded in $\$_n$ by Cayley representation.

**Theorem 6** *Let $G$ be a non-trivial group of order $n$, that is, a subgroup of $\$_n$ by Cayley representation. If $V$ is an $m$-unitary space, $m \geq 2$, and $\chi \in \mathrm{I}(G)$ such that $\chi(1)^2 > |G|/2$, then $V_\chi^n(G)$ does not have an orthogonal basis of decomposable symmetrized tensors.*

*Proof.* Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $V$. Suppose $V_\chi^n(G)$ has an orthogonal basis of decomposable symmetrized tensors. Then, by (3), for all $\gamma \in \overline{\Delta}$, the orbital subspace $V_\gamma^*$ has an orthogonal basis of decomposable symmetrized tensors. Put $\gamma = (1, 2, \ldots, 2) \in \Gamma_m^n$, then we have $G_\gamma = \{1\}$. Therefore, $\sum_{g \in G_\gamma} \chi(g) = \chi(1) \neq 0$, so we can assume that $\gamma \in \overline{\Delta}$.

Therefore, by the above discussion, $V_\gamma^*$, $\gamma = (1, 2, \ldots, 2)$ has an orthogonal basis of decomposable symmetrized tensors. Since $\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) = \chi(1)^2 = s$, we can assume that $\{e^*_{g_1.\gamma}, e^*_{g_2.\gamma}, \ldots, e^*_{g_s.\gamma}\}$ is an orthogonal basis of decomposable symmetrized tensors for $V_\gamma^*$.

Define the $n \times n$ complex matrix $A = [a_{ij}]$ as below:

$$a_{ij} = \frac{\chi(1)}{n} \chi(g_i g_j^{-1}),$$

where $G = \{g_1, \ldots, g_n\}$. For all $i, j$, $1 \leq i, j \leq s$, by (5), we obtain

$$\begin{aligned}
a_{ij} &= \tfrac{\chi(1)}{n} \chi(g_i g_j^{-1}) \\
&= \tfrac{\chi(1)}{|G|} \sum_{\sigma \in G_\gamma} \chi(g_i \sigma g_j^{-1}) \\
&= \langle e^*_{g_j.\gamma} \mid e^*_{g_i.\gamma} \rangle \\
&= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{n} \chi(1) & \text{if } i = j, \end{cases} \\
&= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{s}{n} & \text{if } i = j. \end{cases}
\end{aligned}$$

Therefore, $A$ has the form

$$A = \begin{bmatrix} \frac{s}{n} I_s & A_1 \\ A_2 & A_3 \end{bmatrix},$$

where $A_1, A_2$, and $A_3$ are matrices of sizes $s\times(n-s), (n-s)\times s$, and $(n-s)\times(n-s)$, respectively, and $I_s$ is the $s\times s$ identity matrix. If $A^2=[b_{ij}]$, then by the generalized orthogonality relations for characters, we obtain

$$\begin{aligned}
b_{ij} &= \sum_{k=1}^{n} a_{ik}a_{kj} \\
&= \frac{\chi(1)^2}{n^2}\sum_{k=1}^{n}\chi(g_ig_k^{-1})\chi(g_kg_j^{-1}) \\
&= \frac{\chi(1)^2}{n^2}\sum_{g\in G}\chi(g)\chi(g^{-1}g_ig_j^{-1}) \\
&= \frac{\chi(1)^2}{n^2}\frac{n}{\chi(1)}\chi(g_ig_j^{-1}) \\
&= \frac{\chi(1)}{n}\chi(g_ig_j^{-1}) \\
&= a_{ij}.
\end{aligned}$$

This leads to $A^2=A$ and therefore if we apply this condition for the block form of $A$, we obtain $A_1A_2=(\frac{s}{n}-\frac{s^2}{n^2})I_s$. Since $s\neq n$, $A_1A_2$ is an invertible matrix, and thereby we can easily obtain $s\leq n-s$ or $s\leq\frac{n}{2}$ or $\chi(1)^2\leq|G|/2$ which is a contradiction. Thus, $V_\chi^n(G)$ does not have an orthogonal basis of decomposable symmetrized tensors. $\square$

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